$$= \frac{(2n+1)!}{2(n+1)(n!)^2} \cdot \frac{1}{2^{2n}} = \frac{(2n+2)(2n+1)!}{4(n+1)(n+1)(n!)^2} \cdot \frac{1}{2^{2n}} = \frac{(2n+2)!}{((n+1)!)^2} \cdot \frac{1}{2^{2n+2}}$$

Conclusion : $\forall n \in \mathbf{N},\ U_{2n} = \frac{(2n)!}{(n!)^2} \cdot \frac{1}{2^{2n}}$.

2°/ On a : $\begin{cases} V_{n+1} = (n+2)U_{n+1} \cdot U_{n+2} \\ U_{n+2} = \frac{n+1}{n+2} U_n \end{cases} \Rightarrow V_{n+1} = (n+1)U_{n+1} \cdot U_{2n} = V_n$

donc $\forall n \in \mathbf{N},\ V_{n+1} = V_n$ et par suite (V_n) est une suite constante.

3°/ On a : $\forall n \in \mathbf{N},\ V_n = V_0 = 2 \Rightarrow U_n \cdot U_{n+1} = \frac{2}{n+1}$

$\Rightarrow U_{2n} \cdot U_{2n+1} = \frac{2}{n+1}$ or $U_{2n} = \frac{(2n)!}{(n!)^2} \cdot \frac{1}{2^{2n}}$

$\Rightarrow U_{2n+1} = \frac{2}{(2n+1)} \cdot \frac{(n!)^2 \cdot 2^{2n}}{(2n)!} \Rightarrow U_{2n+1} = \frac{2^{2n+1} \cdot (n!)^2}{(2n+1)!}$.

Solution 10 :

1°/ $U_1 = \sum_{k=0}^{1} \frac{1}{1+k} = 1 + \frac{1}{2} = \frac{3}{2}$; $U_2 = \sum_{k=0}^{2} \frac{2}{8+k} = \frac{2}{8} + \frac{2}{9} + \frac{2}{10} = \frac{121}{180}$.

2°/ On a : $0 \le k \le n \Rightarrow n^3 \le n^3 + k \le n^3 + n \Rightarrow \frac{1}{n^3+n} \le \frac{1}{n^3+k} \le \frac{1}{n^3}$

$\Rightarrow \frac{n}{n^3+n} \le \frac{n}{n^3+k} \le \frac{1}{n^2} \Rightarrow \sum_{k=0}^{n} \frac{n}{n^3+n} \le \sum_{k=0}^{n} \frac{n}{n^3+k} \le \sum_{k=0}^{n} \frac{1}{n^2} \Rightarrow$

$\frac{n(n+1)}{n^3+n} \le U_n \le \frac{n+1}{n^2}$ d'où $\forall n \in \mathbf{N}^*,\ \frac{n+1}{n^2+1} \le U_n \le \frac{n+1}{n^2}$

3°/ $\lim\limits_{n \to +\infty} \frac{n+1}{n^2+1} = \lim\limits_{n \to +\infty} \frac{1}{n} = 0$ et $\lim\limits_{n \to +\infty} \frac{n+1}{n^2} = \lim\limits_{n \to +\infty} \frac{1}{n} = 0$

donc (U_n) converge vers 0.